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2004 J. Phys. A: Math. Gen. 37 4557

(http://iopscience.iop.org/0305-4470/37/16/006)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 4557–4565 PII: S0305-4470(04)74795-2

Determinant structure of R_I type discrete integrable system

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Received 19 January 2004, in final form 11 March 2004
Published 5 April 2004
Online at stacks.iop.org/JPhysA/37/4557 (DOI: 10.1088/0305-4470/37/16/006)

Abstract

A determinant structure of the R_I type discrete integrable system by Vinet–Zhedanov on a semi-infinite lattice is studied using the bilinear method. Bilinear equations of the R_I type discrete integrable system are derived by applying appropriate dependent variable transformations. It is shown that a particular solution for the bilinear equations on a semi-infinite lattice is given in terms of Casorati-type determinants. It is also discussed how the R_I type discrete integrable system relates to the discrete relativistic Toda lattice.

PACS numbers: 02.30.Ik, 02.30.Gp, 05.45.Yv

1. Introduction

The relativistic Toda lattice (RTL) is one of the fundamental models in integrable systems [10]. As is well known, the RTL is reduced to the Toda lattice (TL) in some limit of a parameter. A discrete analogue of the relativistic Toda lattice (dRTL) was proposed by Suris [12]. The dRTL is reduced not only to the RTL, but also to a discrete analogue of the TL, the discrete Toda lattice (dTL). Thus the dRTL can be regarded as a generalization of the RTL, the dTL and the TL.

Recently a new discrete integrable system was derived by Vinet–Zhedanov [13] through the study of spectral transformations for the R_I rational functions. We call this R_I type discrete integrable system the R_I chain. The R_I rational functions, which were introduced by Ismail–Masson [4] in relation to the multi-point Padé approximation, are a generalization of the Laurent biorthogonal polynomials. It is known that the time evolution of the dRTL describes a spectral transformation for the Laurent biorthogonal polynomials [5]. Then it is expected that the R_I chain is a generalization of the dRTL.

The τ function is a fundamental object in the theory of integrable systems, because it reveals the essential features of integrable systems such as Lax representation, Bäcklund transformation, N-soliton solution and so on. By Hirota's bilinear method, integrable systems are transformed to bilinear equations. When the τ functions are expressed as determinants, the bilinear equations are reduced to some determinant identities [1–3, 8]. In this paper, we study a determinant structure of the R_I chain using the bilinear method.

The aim of this paper is to derive bilinear equations of the R_I chain by applying appropriate dependent variable transformations and to clarify a determinant structure of a particular solution for the R_I chain on a semi-infinite lattice using the bilinear method. This paper is organized as follows. In section 2, we review how the R_I chain is derived from the R_I rational functions. In section 3, we derive bilinear equations of the R_I chain and show that a particular solution for the bilinear equations on a semi-infinite lattice is given in terms of Casorati-type determinants. In section 4, we discuss how the R_I chain relates to the dRTL. Section 5 is devoted to concluding remarks.

2. Derivation of the R_I chain

Consider polynomials $P_n(x)$ generated by the recurrence relation

$$P_{n+1}(x) + (u_n x + v_n)P_n(x) + w_n(x - \alpha_n)P_{n-1}(x) = 0 \qquad n = 0, 1, \ldots \tag{1}$$

with the initial conditions

$$P_{-1}(x) = 0 \qquad P_0(x) = 1 \tag{2}$$

where u_n, v_n and w_n are some constants. The R_I rational functions $R_n(x)$ are defined as

$$R_n(x) = \frac{P_n(x)}{\prod_{i=1}^{n}(x - \alpha_i)} \qquad n = 1, 2, \ldots \qquad R_{-1}(x) = 0 \qquad R_0(x) = 1 \tag{3}$$

under the assumptions

$$w_n \neq 0 \qquad P_n(\alpha_n) \neq 0 \qquad n = 1, 2, \ldots. \tag{4}$$

These functions satisfy the recurrence relation

$$(x - \alpha_{n+1})R_{n+1}(x) + (u_n x + v_n)R_n(x) + w_n R_{n-1}(x) = 0 \qquad n = 0, 1, \ldots \tag{5}$$

with the same initial conditions

$$R_{-1}(x) = 0 \qquad R_0(x) = 1. \tag{6}$$

Ismail and Masson established the orthogonality relation for the R_I rational functions.

Theorem 1 (Ismail–Masson [4]). *There exists a linear functional $\mathscr{L}$ on the space of rational functions $x^l / \prod_{i=1}^{k}(x - \alpha_i)$, $k, l = 0, 1, \ldots$, such that the orthogonality relation*

$$\mathscr{L}[R_n(x)x^m] = 0 \qquad m = 0, 1, \ldots, n - 1 \tag{7}$$

holds.

A transformation for the R_I rational functions is given by

$$\tilde{R}_n(x) = \frac{A_n(x - \alpha_{n+1})R_{n+1}(x) + B_n R_n(x)}{x - \lambda} \tag{8}$$

where A_n and B_n are some constants satisfying the relation

$$A_n R_{n+1}(\lambda) = B_n R_n(\lambda). \tag{9}$$

It is easily shown that the new rational functions $\tilde{R}_n(x)$ are again the R_I rational functions satisfying the orthogonality relation

$$\tilde{\mathscr{L}}[\tilde{R}_n(x)x^m] = 0 \qquad m = 0, 1, \ldots, n - 1 \tag{10}$$

where

$$\tilde{\mathscr{L}} = (x - \lambda)\mathscr{L}. \tag{11}$$

The transformation (8) is called the Christoffel transformation. There also exists the reciprocal transformation to the Christoffel transformation

$$R_n(x) = \tilde{R}_n(x) + C_n \tilde{R}_{n-1}(x) \tag{12}$$

where C_n is some constant. This transformation (12) is called the Geronimus transformation. These transformations (8) and (12) can be regarded as spectral transformations for the R_I rational functions [13].

The spectral transformations (8) and (12) for the R_I rational functions induce a discrete integrable system which the coefficients A_n, B_n and C_n satisfy. To see this, we introduce a discrete time t as the number of times that the Christoffel transformations are applied to the initial R_I rational functions $R_n^0(x) = R_n(x)$. Then the spectral transformations are written as follows:

$$R_n^{t+1}(x) = \frac{A_n^t(x - \alpha_{n+1})R_{n+1}^t(x) + B_n^t R_n^t(x)}{x - \lambda_t} \tag{13a}$$

$$R_n^t(x) = R_n^{t+1}(x) + C_n^t R_{n-1}^{t+1}(x). \tag{13b}$$

From the compatibility condition of the spectral transformations (13) we derive a discrete integrable system, the R_I chain,

$$\frac{A_{n-1}^{t+1}C_n^{t+1} - 1}{A_n^{t+1}} = \frac{A_n^t C_{n+1}^t - 1}{A_n^t} \tag{14a}$$

$$\frac{\alpha_n A_{n-1}^{t+1}C_n^{t+1} - B_n^{t+1} - \lambda_{t+1}}{A_n^{t+1}} = \frac{\alpha_{n+1}A_n^t C_{n+1}^t - B_n^t - \lambda_t}{A_n^t} \tag{14b}$$

$$\frac{B_{n-1}^{t+1}C_n^{t+1}}{A_n^{t+1}} = \frac{B_n^t C_n^t}{A_n^t}. \tag{14c}$$

We note that the R_I chain (14) is essentially the same as the discrete integrable system derived in [13].

3. Determinant solution on the semi-infinite lattice

Applying the dependent variable transformations

$$A_n^{k,t} = -\frac{\tau_n^{k+1,t+1}\tau_{n+1}^{k,t}}{\tau_n^{k,t+1}\tau_{n+1}^{k+1,t}} \tag{15a}$$

$$B_n^{k,t} = \frac{\tau_n^{k,t}\tau_{n+1}^{k+1,t+1}}{\tau_n^{k,t+1}\tau_{n+1}^{k+1,t}} \tag{15b}$$

$$C_n^{k,t} = \frac{\tau_{n-1}^{k,t+1}\tau_{n+1}^{k,t}}{\tau_n^{k,t}\tau_n^{k,t+1}} \tag{15c}$$

the R_I chain (14) is transformed to the bilinear equations

$$\tau_n^{k,t}\tau_n^{k+1,t+1} - \tau_n^{k,t+1}\tau_n^{k+1,t} = \tau_{n-1}^{k+1,t+1}\tau_{n+1}^{k,t} \tag{16a}$$

$$\tau_n^{k,t}\tau_{n+1}^{k,t+1} + (\lambda_t - \alpha_{k+n})\tau_n^{k,t+1}\tau_{n+1}^{k,t} = \tau_n^{k+1,t+1}\tau_{n+1}^{k-1,t} \tag{16b}$$

$$\tau_n^{k-1,t}\tau_n^{k+1,t} + \tau_{n-1}^{k,t}\tau_{n+1}^{k,t} = \tau_n^{k,t}\sigma_n^{k,t} \tag{16c}$$

$$\tau_n^{k+1,t}\sigma_{n+1}^{k,t} - \tau_{n+1}^{k,t}\sigma_n^{k+1,t} = (\alpha_{k+n+1} - \alpha_{k+n})\tau_n^{k,t}\tau_{n+1}^{k+1,t} \tag{16d}$$

where the index k denotes an auxiliary independent variable. It can be shown that if $\tau_n^{k,t}$ and $\sigma_n^{k,t}$ satisfy the bilinear equations (16), then $A_n^{k,t}$, $B_n^{k,t}$ and $C_n^{k,t}$ satisfy the R_I chain (14).

We give a solution for the R_I chain on a semi-infinite lattice.

Theorem 2. *Define the τ functions $\tau_n^{k,t}$ and $\sigma_n^{k,t}$ as*

$$\tau_n^{k,t} = \begin{vmatrix} c_{k,0}^t & c_{k,1}^t & \cdots & c_{k,n-1}^t \\ c_{k+1,0}^t & c_{k+1,1}^t & \cdots & c_{k+1,n-1}^t \\ \vdots & \vdots & & \vdots \\ c_{k+n-1,0}^t & c_{k+n-1,1}^t & \cdots & c_{k+n-1,n-1}^t \end{vmatrix} \tag{17a}$$

$$\sigma_n^{k,t} = \begin{vmatrix} c_{k+n,n}^t - \alpha_{k+n-1}c_{k+n,n-1}^t & c_{k+n-2,n-1}^t & \cdots & c_{k+n-2,2n-3}^t \\ c_{k+n,n-1} - \alpha_{k+n-1}c_{k+n,n-2}^t & c_{k+n-2,n-2}^t & \cdots & c_{k+n-2,2n-4}^t \\ \vdots & \vdots & & \vdots \\ c_{k+n,1}^t - \alpha_{k+n-1}c_{k+n,0}^t & c_{k+n-2,0}^t & \cdots & c_{k+n-2,n-2}^t \end{vmatrix} \tag{17b}$$

where the element $c_{k,l}^t$ satisfies the dispersion relations

$$c_{k-1,l}^t = c_{k,l+1}^t - \alpha_k c_{k,l}^t \tag{18a}$$

$$c_{k,l}^{t+1} = c_{k,l+1}^t - \lambda_t c_{k,l}^t. \tag{18b}$$

Then the τ functions (17) give a solution for the bilinear equations of the R_I chain (16) on the semi-infinite lattice

$$\tau_{-1}^{k,t} = \tau_{-2}^{k,t} = \cdots = 0 \qquad \sigma_{-1}^{k,t} = \sigma_{-2}^{k,t} = \cdots = 0 \qquad \tau_0^{k,t} = \sigma_0^{k,t} = 1. \tag{19}$$

In terms of the variables $A_n^{k,t}$, $B_n^{k,t}$ and $C_n^{k,t}$, the corresponding boundary condition is given by

$$A_{-1}^{k,t} = A_{-2}^{k,t} = \cdots = 0 \qquad B_{-1}^{k,t} = B_{-2}^{k,t} = \cdots = 0 \qquad C_0^{k,t} = C_{-1}^{k,t} = \cdots = 0. \tag{20}$$

Proof. We show that the τ functions (17) satisfy the bilinear equations (16).

Let D be some determinant, and $D\left[\begin{smallmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{smallmatrix}\right]$ be the determinant with the $i_1, \ldots, i_n$th rows and the $j_1, \ldots, j_n$th columns removed from D. Then the following identity is satisfied,

$$D \cdot D\begin{bmatrix} i & k \\ j & l \end{bmatrix} = D\begin{bmatrix} i \\ j \end{bmatrix} D\begin{bmatrix} k \\ l \end{bmatrix} - D\begin{bmatrix} i \\ l \end{bmatrix} D\begin{bmatrix} k \\ j \end{bmatrix} \tag{21}$$

which is called the Jacobi identity. The bilinear equation (16a) follows from the Jacobi identity (21) with $i = j = 1$, $k = l = n + 1$, where D is given by

$$D = \begin{vmatrix} c_{k,0}^t & c_{k,0}^{t+1} & \cdots & c_{k,n-2}^{t+1} & c_{k,n-1}^{t+1} \\ c_{k+1,0}^t & c_{k+1,0}^{t+1} & \cdots & c_{k+1,n-2}^{t+1} & c_{k+1,n-1}^{t+1} \\ \vdots & \vdots & & \vdots & \vdots \\ c_{k+n-1,0}^t & c_{k+n-1,0}^{t+1} & \cdots & c_{k+n-1,n-2}^{t+1} & c_{k+n-1,n-1}^{t+1} \\ c_{k+n,0}^t & c_{k+n,0}^{t+1} & \cdots & c_{k+n,n-2}^{t+1} & c_{k+n,n-1}^{t+1} \end{vmatrix}. \tag{22}$$

Indeed, we can easily see that

$$D = \tau_{n+1}^{k,t} \tag{23a}$$

$$D\begin{bmatrix} 1 & n+1 \\ 1 & n+1 \end{bmatrix} = \tau_{n-1}^{k+1,t+1} \tag{23b}$$

$$D\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \tau_{n}^{k+1,t+1} \tag{23c}$$

$$D\begin{bmatrix} n+1 \\ n+1 \end{bmatrix} = \tau_{n}^{k,t} \tag{23d}$$

$$D\begin{bmatrix} 1 \\ n+1 \end{bmatrix} = \tau_{n}^{k+1,t} \tag{23e}$$

$$D\begin{bmatrix} n+1 \\ 1 \end{bmatrix} = \tau_{n}^{k,t+1}. \tag{23f}$$

Similarly, we obtain the bilinear equation (16c) from the Jacobi identity (21) with $i = j = 1$, $k = l = n + 1$ and

$$D = \begin{vmatrix} c_{k+n,n}^t - \alpha_{k+n-1} c_{k+n,n-1}^t & c_{k+n-2,n-1}^t & \cdots & c_{k+n-2,2n-3}^t & c_{k+n-2,2n-2}^t \\ c_{k+n,n-1} - \alpha_{k+n-1} c_{k+n,n-2}^t & c_{k+n-2,n-2}^t & \cdots & c_{k+n-2,2n-4}^t & c_{k+n-2,2n-3}^t \\ \vdots & \vdots & & \vdots & \vdots \\ c_{k+n,1}^t - \alpha_{k+n-1} c_{k+n,0}^t & c_{k+n-2,0}^t & \cdots & c_{k+n-2,n-2}^t & c_{k+n-2,n-1}^t \\ c_{k+n,0}^t & c_{k+n-1,0}^t & \cdots & c_{k+n-1,n-2}^t & c_{k+n-1,n-1}^t \end{vmatrix}. \tag{24}$$

We have now proved that the τ functions (17) satisfy the bilinear equations (16a), (16c).

Next, we move on to the proofs that the bilinear equations (16b), (16d) are satisfied by the τ functions (17). Consider the identity

$$\left|\begin{array}{ccc|c|ccc|cc} f_1 & \cdots & f_n & a_1 & & \emptyset & & a_2 & a_3 \\ \hline & \emptyset & & a_1 & f_1 & \cdots & f_{n-1} & a_2 & a_3 \end{array}\right| = 0 \tag{25}$$

where f_i, a_i are arbitrary $(n+1)$-dimensional column vectors. Applying the Laplace expansion to the left-hand side of identity (25), we obtain

$$\begin{aligned} &|f_1 \ \cdots \ f_{n-1} \ f_n \ a_1| \cdot |f_1 \ \cdots \ f_{n-1} \ a_2 \ a_3| \\ &-|f_1 \ \cdots \ f_{n-1} \ f_n \ a_2| \cdot |f_1 \ \cdots \ f_{n-1} \ a_1 \ a_3| \\ &+|f_1 \ \cdots \ f_{n-1} \ f_n \ a_3| \cdot |f_1 \ \cdots \ f_{n-1} \ a_1 \ a_2| = 0 \end{aligned} \tag{26}$$

which is one of the Plücker relations. The bilinear equation (16b) follows from the Plücker relation (26) with

$$f_i = \begin{pmatrix} c_{k+n-1,n+i-1}^{t+1} & \cdots & c_{k+n-1,i-1}^{t+1} \end{pmatrix}^\top \tag{27a}$$

$$a_1 = \begin{pmatrix} c_{k+n,n}^{t+1} & \cdots & c_{k+n,0}^{t+1} \end{pmatrix}^\top \tag{27b}$$

$$a_2 = \begin{pmatrix} c_{k+n-1,n}^{t} & \cdots & c_{k+n-1,0}^{t} \end{pmatrix}^\top \tag{27c}$$

$$a_3 = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}^\top. \tag{27d}$$

Indeed, we can see that

$$|f_1 \quad \cdots \quad f_{n-1} \quad f_n \quad a_1| = (-1)^n \tau_{n+1}^{k,t+1} \tag{28a}$$

$$|f_1 \quad \cdots \quad f_{n-1} \quad a_2 \quad a_3| = -\tau_n^{k,t} \tag{28b}$$

$$|f_1 \quad \cdots \quad f_{n-1} \quad f_n \quad a_2| = (-1)^n \tau_{n+1}^{k-1,t} \tag{28c}$$

$$|f_1 \quad \cdots \quad f_{n-1} \quad a_1 \quad a_3| = -\tau_n^{k+1,t+1} \tag{28d}$$

$$|f_1 \quad \cdots \quad f_{n-1} \quad f_n \quad a_3| = (-1)^n \tau_n^{k,t+1} \tag{28e}$$

$$|f_1 \quad \cdots \quad f_{n-1} \quad a_1 \quad a_2| = -(\lambda_t - \alpha_{k+n}) \tau_{n+1}^{k,t}. \tag{28f}$$

Similarly, the bilinear equation (16d) is reduced to the Plücker relation (26) with

$$f_i = \left(c_{k+n-1,n+i-1}^{t+1} \quad \cdots \quad c_{k+n-1,i-1}^{t+1}\right)^\top \tag{29a}$$

$$a_1 = \left(c_{k+n,n}^{t} \quad \cdots \quad c_{k+n,0}^{t}\right)^\top \tag{29b}$$

$$a_2 = \left(c_{k+n+1,n+1}^{t} - \alpha_{k+n} c_{k+n+1,n}^{t} \quad \cdots \quad c_{k+n+1,1}^{t} - \alpha_{k+n} c_{k+n+1,0}^{t}\right)^\top \tag{29c}$$

$$a_3 = (1 \quad 0 \quad \cdots \quad 0)^\top. \tag{29d}$$

This completes the proof. □

4. Reduction to the discrete relativistic Toda lattice

As was stated in section 2, the R_I chain is derived from the compatibility condition of the spectral transformations for the R_I rational functions. The R_I rational functions are reduced to the Laurent biorthogonal polynomials in the case where all the poles of the R_I rational functions are equal to some constant: $\alpha_k = \alpha$ for all k. Indeed, we obtain the recurrence relation of the Laurent biorthogonal polynomials by shifting the argument x to $x + \alpha$ in the recurrence relation of the R_I rational functions (5). It is known that the spectral transformations for the Laurent biorthogonal polynomials induce the dRTL [5]. In this section, we discuss how the R_I chain relates to the dRTL.

The dRTL proposed by Suris [12] is given by

$$\frac{\delta \exp\left(q_n^{t+1} - q_n^t\right) - 1}{\delta \exp\left(q_n^t - q_n^{t-1}\right) - 1} = \frac{1 + g^2 \exp\left(q_{n-1}^t - q_n^t\right)}{1 + g^2 \exp\left(q_n^t - q_{n+1}^t\right)} \frac{1 + (g^2/\delta) \exp\left(q_n^t - q_{n+1}^{t+1}\right)}{1 + (g^2/\delta) \exp\left(q_{n+1}^{t-1} - q_n^t\right)} \tag{30}$$

where $\delta = \exp(ch)$, c is the speed of light, h is a difference interval, and g is a coupling constant. This system includes several integrable systems as special cases [6, 12]. For example, taking the continuous limit $h \to 0$ with $g^2 c^2 = 1$, we obtain from the dRTL (30) the RTL [10]

$$\begin{aligned} \frac{\mathrm{d}^2 q_n(t)}{\mathrm{d}t^2} &= \left(c + \frac{\mathrm{d}q_n(t)}{\mathrm{d}t}\right)\left(c + \frac{\mathrm{d}q_{n+1}(t)}{\mathrm{d}t}\right) \frac{g^2 \exp(q_{n+1}(t) - q_n(t))}{1 + g^2 \exp(q_{n+1}(t) - q_n(t))} \\ &\quad - \left(c + \frac{\mathrm{d}q_{n-1}(t)}{\mathrm{d}t}\right)\left(c + \frac{\mathrm{d}q_n(t)}{\mathrm{d}t}\right) \frac{g^2 \exp(q_n(t) - q_{n-1}(t))}{1 + g^2 \exp(q_n(t) - q_{n-1}(t))} \end{aligned} \tag{31}$$

which is reduced to the TL

$$\frac{\mathrm{d}^2 q_n(t)}{\mathrm{d}t^2} = \exp(q_{n+1}(t) - q_n(t)) - \exp(q_n(t) - q_{n-1}(t)) \tag{32}$$

in the limit of $c \to \infty$. Taking the limit $c \to \infty$ with $g^2/h^2 = 1$, we also obtain the dTL

$$\exp\left(q_n^{t+1} - 2q_n^t + q_n^{t-1}\right) = \frac{1 + g^2 \exp\left(q_{n+1}^t - q_n^t\right)}{1 + g^2 \exp\left(q_n^t - q_{n-1}^t\right)} \tag{33}$$

which is reduced to the TL (32) in the continuous limit of $h \to 0$.

In the case where all the α_k are equal to some constant, the R_I chain is reduced to the dRTL. In the subsequent discussion, we show this in terms of the bilinear equations. The dRTL (30) is transformed to the bilinear equations

$$\tau_n^{k-1,t}\tau_n^{k+1,t} + \xi\tau_{n-1}^{k,t}\tau_{n+1}^{k,t} = \left(\tau_n^{k,t}\right)^2 \tag{34a}$$

$$\xi\tau_n^{k,t}\tau_{n+1}^{k,t+1} + \xi\eta(\lambda-\alpha)\tau_n^{k,t+1}\tau_{n+1}^{k,t} = \eta\tau_n^{k+1,t+1}\tau_{n+1}^{k-1,t} \tag{34b}$$

$$\tau_n^{k,t}\tau_n^{k,t+1} + \xi\eta(\lambda-\alpha)\tau_{n-1}^{k,t+1}\tau_{n+1}^{k,t} = \tau_n^{k-1,t}\tau_n^{k+1,t+1} \tag{34c}$$

through the variable transformations

$$\exp q_n^t = \frac{\tau_n^{k,t}}{\tau_{n-1}^{k,t}} \tag{35a}$$

$$g^2 = -\xi \tag{35b}$$

$$\delta = -\frac{1}{\eta(\lambda-\alpha)}. \tag{35c}$$

These bilinear equations and their particular solutions are discussed in [6, 7]. See also [9] for bilinear equations of the RTL (31) and their particular solutions.

We introduce multiplier factors to the dispersion relations of the τ functions of the R_I chain (18) as

$$c_{k-1,l}^t = \xi_k\left(c_{k,l+1}^t - \alpha_k c_{k,l}^t\right) \tag{36a}$$

$$c_{k,l}^{t+1} = \eta_t\left(c_{k,l+1}^t - \lambda_t c_{k,l}^t\right). \tag{36b}$$

Thus the bilinear equations of the R_I chain (16) are modified as follows:

$$\tau_n^{k,t}\tau_n^{k+1,t+1} - \tau_n^{k,t+1}\tau_n^{k+1,t} = \eta_t\tau_{n-1}^{k+1,t+1}\tau_{n+1}^{k,t} \tag{37a}$$

$$\xi_k\tau_n^{k,t}\tau_{n+1}^{k,t+1} + \xi_k\eta_t(\lambda_t - \alpha_{k+n})\tau_n^{k,t+1}\tau_{n+1}^{k,t} = \eta_t\tau_n^{k+1,t+1}\tau_{n+1}^{k-1,t} \tag{37b}$$

$$\tau_n^{k-1,t}\tau_n^{k+1,t} + \xi_k\tau_{n-1}^{k,t}\tau_{n+1}^{k,t} = \xi_k\xi_{k+1}(\xi_{k+2})^2\cdots(\xi_{k+n-2})^{n-2}\tau_n^{k,t}\sigma_n^{k,t} \tag{37c}$$

$$\tau_n^{k+1,t}\sigma_{n+1}^{k,t} - \xi_{k+1}\cdots\xi_{k+n-1}\tau_{n+1}^{k,t}\sigma_n^{k+1,t} = \frac{\alpha_{k+n+1}-\alpha_{k+n}}{\xi_{k+1}(\xi_{k+2})^2\cdots(\xi_{k+n-1})^{n-1}}\tau_n^{k,t}\tau_{n+1}^{k+1,t}. \tag{37d}$$

Here setting $\alpha_k = \alpha$ for all k, the four bilinear equations (37) are reduced to the three bilinear equations

$$\tau_n^{k,t}\tau_n^{k+1,t+1} - \tau_n^{k,t+1}\tau_n^{k+1,t} = \eta_t\tau_{n-1}^{k+1,t+1}\tau_{n+1}^{k,t} \tag{38a}$$

$$\xi_k\tau_n^{k,t}\tau_{n+1}^{k,t+1} + \xi_k\eta_t(\lambda_t - \alpha)\tau_n^{k,t+1}\tau_{n+1}^{k,t} = \eta_t\tau_n^{k+1,t+1}\tau_{n+1}^{k-1,t} \tag{38b}$$

$$\xi_{k+n}\tau_n^{k-1,t}\tau_n^{k+1,t} + \xi_k\xi_{k+n}\tau_{n-1}^{k,t}\tau_{n+1}^{k,t} = \xi_k\left(\tau_n^{k,t}\right)^2. \tag{38c}$$

These bilinear equations (38) with $\lambda_t = \lambda$, $\xi_k = \xi$, $\eta_t = \eta$ for all k and t are equivalent to the bilinear equations of the discrete relativistic Toda lattice (34). Equations (38*b*) and (38*c*) go to equations (34*a*) and (34*b*) respectively. Multiplying both sides of equation (38*a*) by $\tau_{n-1}^{k,t+1}$ and using equation (38*c*), we obtain

$$\xi_k \tau_n^{k,t} \tau_n^{k,t+1} + \xi_k \xi_{k+1} \eta_t (\lambda_t - \alpha) \tau_{n-1}^{k,t+1} \tau_{n+1}^{k,t} = \xi_{k+1} \tau_n^{k-1,t} \tau_n^{k+1,t+1} \tag{39}$$

which goes to equation (34*c*).

5. Concluding remarks

In this paper, we have derived bilinear equations of the R_I chain and shown that a particular solution for the R_I chain on a semi-infinite lattice is given in terms of Casorati-type determinants. We have also discussed how the R_I chain relates to the dRTL. As a result, we have obtained the following hierarchical diagram:

$$\begin{array}{ccccc} R_I \text{ chain} & \longrightarrow & \text{dRTL} & \longrightarrow & \text{dTL} \\ & & \downarrow & & \downarrow \\ & & \text{RTL} & \longrightarrow & \text{TL} \end{array}$$

In comparison with the dTL, a special feature of the R_I chain is that it has two arbitrary parameters, the time-dependent parameter λ_t and the space-dependent parameter α_n. The dTL has only a time-dependent parameter. Of the two parameters, λ_t denotes a nonuniform difference interval of the discrete time t, which is the same situation as in the case of the dTL. On the other hand, the space-dependent parameter α_n does not have such a interpretation in the original form of the R_I chain. In order to derive bilinear equations of the R_I chain, we have introduced the auxiliary independent variable k to the R_I chain and generalized it. The space-dependent parameter α_n can now be regarded as a nonuniform difference interval of k.

We have considered a solution for the R_I chain on a semi-infinite lattice in this paper. It is also an important problem to derive solutions on other lattices such as an infinite lattice or a periodic lattice.

As we mentioned in section 2, the R_I chain was derived through the study of the R_I rational functions. There exists a more general class including the R_I rational functions, which is called the R_{II} rational functions [4]. Spiridonov–Zhedanov [11] derived a discrete integrable system called the R_{II} chain from a compatibility condition of spectral transformations for the R_{II} chain. The R_{II} chain has three arbitrary parameters, one more arbitrary parameter than the R_I chain. It is an interesting problem to study the R_{II} chain using the bilinear method.

Acknowledgments

The authors are grateful to Y Nakamura and A Zhedanov for many fruitful discussions and helpful advice. This paper is supported in part by Grant-in-Aid for Scientific Research (no 15005102, 15540119) from the Ministry of Education, Culture, Sports, Science and Technology, Japan.

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